

Moments of generalized Husimi distributions and complexity of many-body quantum states

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Abstract

We define generalized Husimi distributions using generalized coherent states, and show that their moments are good measures of complexity of many-body quantum states. Our construction of the coherent states is based on the single-particle transformation group of the system. Then the coherent states are independent-particle states, and, at the same time, the most localized states in the Husimi representation. Therefore delocalization of the Husimi distribution, which can be measured by the moments, is a sign of many-body correlation (entanglement). Since the delocalization of the Husimi distribution is also related to chaoticity of the dynamics, it suggests a relation between entanglement and chaos. Our definition of the Husimi distribution can be applied not only to the systems of distinguishable particles, but also to those of identical particles, i.e., fermions and bosons. We derive an algebraic formula to evaluate the moments of the Husimi distribution.

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I. INTRODUCTION

Although the manifestation of chaos in quantum systems has been investigated extensively in the past few decades, definition of quantum chaos is still unclear. Especially the studies of many-body systems are far behind those of one-body systems.

In [1], we proposed a measure of complexity (chaoticity) of quantum states in one-body systems by using the Husimi distribution [2]. The purpose of this paper is to generalize this idea to many-body problems. We will show that the generalized Husimi distribution is also related to quantum many-body correlation (entanglement). Therefore it implies a relation between entanglement and chaoticity of the dynamics.

In one-body systems, the Husimi distribution function of a quantum state $|\varphi\rangle$ is defined as [17]

$$\mathcal{H}_{|\varphi\rangle}(\mathbf{p}, \mathbf{q}) \equiv |\langle \mathbf{p}, \mathbf{q} | \varphi \rangle|^2, \quad (1)$$

where $|\mathbf{p}, \mathbf{q}\rangle$ is a coherent state whose average momentum and coordinate are respectively \mathbf{p} and \mathbf{q} . Since chaoticity of classical systems can be characterized by delocalization of trajectories, delocalization of the Husimi distribution can be regarded as a quantum manifestation of chaos. We used the second moment as a simple measure of the delocalization. The inverse of the second moment represents the effective volume occupied by the Husimi distribution, and has a good correspondence with chaoticity of the classical system.

To extend this idea to many-body systems is not difficult, at least formally. If we define coherent states in a many-body system, the Husimi distribution is defined as the square of the absolute value of the coherent state representation. There are several ways to generalize the idea of coherent state to many-body systems. We construct the coherent states based on the single-particle transformation group, following the group-theoretical construction by Perelomov [3]. Then the coherent states are independent-particle states, as we will explain below.

Let us consider a system of m qubits as an example. In this case, the single-particle (local unitary) transformation group is $\overbrace{SU(2) \times \cdots \times SU(2)}^m$. Coherent states are generated by applying this group to the “vacuum” [18] $|0\rangle \otimes \cdots \otimes |0\rangle$. Then we obtain all separable (disentangled) states. Hence,

$$\text{coherent state} \iff \text{disentangled state}. \quad (2)$$

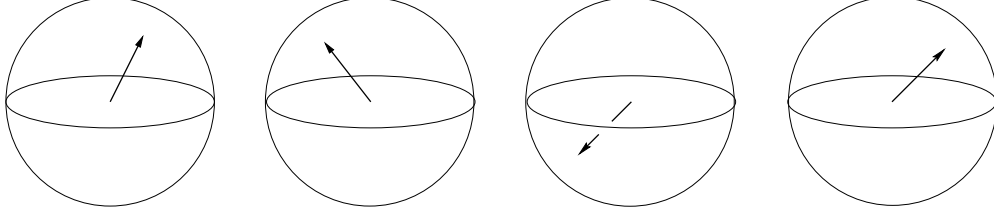


FIG. 1: The Husimi distribution of qubits is defined on $S^2 \times S^2 \times \dots \times S^2$. A disentangled state is represented by a point on this manifold, and the Husimi distribution thereof is localized around the point. An entangled state is represented by a delocalized distribution.

The coherent states are parametrized as

$$|\boldsymbol{\theta}, \boldsymbol{\phi}\rangle = |\theta_1, \phi_1\rangle \otimes \dots \otimes |\theta_m, \phi_m\rangle, \quad (3)$$

where $|\theta, \phi\rangle$ is the Bloch sphere representation of a qubit. The Husimi distribution of a quantum state $|\varphi\rangle$ is defined as

$$\mathcal{H}_{|\varphi\rangle}(\boldsymbol{\theta}, \boldsymbol{\phi}) = |\langle \boldsymbol{\theta}, \boldsymbol{\phi} | \varphi \rangle|^2 \quad (4)$$

on the “phase space” $S^2 \times \dots \times S^2$. A disentangled state is represented by a localized wave packet in the phase space. (Fig. 1)

The Husimi distribution can also be defined for systems of identical particles, i.e., bosons and fermions. Suppose there are m identical particles in N single-particle states. In this case, we cannot operate each particle separately. Therefore the single-particle transformation group of this system is not $\overbrace{SU(N) \times \dots \times SU(N)}^m$, but $SU(N)$ (or $U(N)$) [19]. The coherent states are defined as

$$|\zeta\rangle = U(\zeta)|0\rangle, \quad (5)$$

where $U(\zeta)$ is an element of $U(N)$ specified by the parameter ζ . The “vacuum” $|0\rangle$ can be written explicitly as

$$|0\rangle = \begin{cases} |\varphi_1\rangle|\varphi_1\rangle \dots |\varphi_1\rangle & (\text{boson}) \\ \mathcal{A}(|\varphi_1\rangle|\varphi_2\rangle \dots |\varphi_m\rangle) & (\text{fermion}) \end{cases}, \quad (6)$$

where $|\varphi_i\rangle$ is the i -th single-particle state and \mathcal{A} is the anti-symmetrization operator. Then the coherent state is

$$|\zeta\rangle = \begin{cases} |\varphi_1(\zeta)\rangle|\varphi_1(\zeta)\rangle \dots |\varphi_1(\zeta)\rangle & (\text{boson}) \\ \mathcal{A}(|\varphi_1(\zeta)\rangle|\varphi_2(\zeta)\rangle \dots |\varphi_m(\zeta)\rangle) & (\text{fermion}) \end{cases}, \quad (7)$$

where $|\varphi_i(\zeta)\rangle = U(\zeta)|\varphi_i\rangle$. In the bosonic case, the coherent states are separable, and it is easy to see that all separable bosonic states can be written in this form. Therefore

$$\text{coherent state} \iff \text{separable state.} \quad (8)$$

In the fermionic case, a coherent state is a Slater determinant, and any Slater determinant can be written as a coherent state. Namely,

$$\text{coherent state} \iff \text{Slater determinant.} \quad (9)$$

In any case, the coherent states are the least correlated ones. We summarize the features of these systems in Table. I.

The Husimi distribution is defined by using the coherent state as

$$\mathcal{H}_{|\varphi\rangle}(\zeta) \equiv |\langle\zeta|\varphi\rangle|^2. \quad (10)$$

Since the most localized states in the Husimi representation are the coherent states, delocalization of the Husimi distribution implies correlation among the particles. The delocalization can be measured by the Rényi-Wehrl entropy [4]

$$S_{|\varphi\rangle}^{(q)} \equiv \frac{1}{1-q} \ln M_{|\varphi\rangle}^{(q)}, \quad (11)$$

where $M^{(q)}$ is the moment with an index $q > 0$:

$$M_{|\varphi\rangle}^{(q)} = \int d\mu(\zeta) \{\mathcal{H}_{|\varphi\rangle}(\zeta)\}^q. \quad (12)$$

Here, $d\mu(\zeta)$ is the Haar measure of the group. Hence $M^{(q)}$ is an invariant of the group transformation. Note that $S^{(q)}$ reproduces the normal entropy in the limit $q \rightarrow 1$:

$$\lim_{q \rightarrow 1} S_{|\varphi\rangle}^{(q)} = - \int d\mu(\xi) \mathcal{H}_{|\varphi\rangle}(\xi) \ln \mathcal{H}_{|\varphi\rangle}(\xi). \quad (13)$$

The Rényi-Wehrl entropy represents the effective volume occupied by the Husimi distribution. For instance, if the Husimi distribution takes the same value over a region with volume V and takes zero value outside of it [20], $S^{(q)} = \ln V$. Therefore it is natural to expect that $S^{(q)}$ takes the minimum value for the coherent states. This conjecture (generalized Lieb-Wehrl conjecture) was formed in [4], following the questions raised by Wehrl [5] and Lieb [6]. For integer indices $q \geq 2$ it was proved in [7]. According to the conjecture,

$$S_{|\varphi\rangle}^{(q)} = S_{min}^{(q)} \iff |\varphi\rangle : \begin{cases} \text{separable} & (\text{qubits, bosons}) \\ \text{Slater determinant} & (\text{fermions}) \end{cases}. \quad (14)$$

Thus $S^{(q)}$ shows whether the particles are correlated or not. If $S^{(q)}$ is large, the Husimi distribution is delocalized, and we need a lot of wave packets (independent-particle states) to represent the state. Therefore the Rényi-Wehrl entropy can be regarded as a measure of complexity.

Note that the complexity we measure here is completely independent of the complexity of single-particle states which was the subject of [1]. For instance, we can construct a Slater determinant from highly chaotic single-particle states, which is still the most localized state in the many-body Husimi distribution.

When a Hamiltonian of the system is specified, the time evolution operator of the system can be represented by the coherent state path integral. The stationary phase condition of this integral leads to an equation of motion in the phase space, which defines the “classical” dynamics of the system. (See, for example, [8].) In the case of qubits, for example, the equation of motion is that of classical spins. When the “classical” motion is chaotic, the Husimi distribution will probably spread all over the phase space. This means that chaotic dynamics leads to highly complex states.

Next we consider how to calculate the moments (12). Although the definition of the moments (12) using the Haar measure is an elegant way to obtain invariants of the group, this integral representation is almost useless in numerical calculations because the dimension of the phase space is huge when there are many degrees of freedom. Since the coherent states form an overcomplete set, the coherent state representation is highly redundant. We should avoid such redundancy in real calculations, as we did in [1] for one-body problems.

A complex quantum state is usually represented by expansion coefficients in a basis. Therefore it is desirable to represent the moments (12) directly by them. Fortunately, it is possible if the index q is a positive integer. We will derive an algebraic formula which represents the moment directly by the expansion coefficients in section II. This explicit formula is not only practically useful, but also theoretically insightful.

This paper is organized as follows. In section II we derive the explicit formula for the moments. In this formula, the moments are represented by the expansion coefficients of the state and group-theoretical factors which are independent of the state. The main concern in the following sections is to determine the group-theoretical factors. After explaining the general formalism, we consider bosonic systems with two single-particle states in detail in section III. These systems have $U(2) \simeq U(1) \times SU(2)$ as the single-particle transformation

	one-body system	bosons	fermions	distinguishable particles
	(n dim.)	$B[N, m]$	$F[N, m]$	$D[N_1, \dots, N_m]$
group	HW group	$U(N)$	$U(N)$	$SU(N_1) \times \dots \times SU(N_m)$
phase space	\mathbb{R}^{2n}	$\mathbb{C}P^{N-1}$	$G_{m,N}(\mathbb{C})$	$\mathbb{C}P^{N_1-1} \times \dots \times \mathbb{C}P^{N_m-1}$
coherent state	Gaussian	separable state	Slater Det.	separable state
	wave packet	$ \varphi\rangle \varphi\rangle \dots \varphi\rangle$	$\mathcal{A}(\varphi_1\rangle \varphi_2\rangle \dots \varphi_m\rangle)$	$ \varphi_1\rangle \varphi_2\rangle \dots \varphi_m\rangle$

TABLE I: Coherent states for one-body systems, bosons, fermions and distinguishable particles. $B[N, m]$ ($F[N, m]$) means a system of m bosons (fermions) in N single-particle states. $D[N_1, \dots, N_m]$ is a system of m distinguishable particles in which i -th particles takes N_i states. $\mathbb{C}P^n$ is the n -dimensional complex projective space and $G_{m,N}(\mathbb{C}) \simeq U(N)/(U(N-m) \times U(m))$ is the Grassmann manifold. Note that $\mathbb{C}P^1 \simeq S^2$. HW group in the second row means the Heisenberg-Weyl group, which is generated by \hat{p} and \hat{q} .

group, and the corresponding phase space is the two-dimensional sphere S^2 . Therefore we can visualize the Husimi distribution in this case, which will help us understand the idea of the Husimi distribution in many-body systems. Based on this analysis, we treat bosonic systems generally in section IV. In section V we treat fermionic systems. In section VI, we consider distinguishable particles including qubits. We investigate the second moment in 2- and 3-qubit cases in detail, and show the relation between the second moment and other known measures of entanglement. The final section is devoted to concluding remarks.

II. EXPLICIT FORMULA FOR THE MOMENTS WITH INTEGER INDICES

In this section, we show how to obtain an explicit formula for the moment $M^{(q)}$ for a positive integer q . Our method is based on the expression of the moments (21) using the tensor products of the state, which was used also in [7] to prove the generalized Lieb-Wehrl conjecture. We briefly review the derivation of (21), and show how the formula is obtained from it.

To begin with, we briefly state the definition of the generalized coherent states of a compact semisimple Lie group G [3]. The Lie algebra of G can be written in Cartan basis $\{E_\alpha, H_j\}$. Let D_λ be an irreducible representation space of G with the lowest weight $-\lambda$.

Coherent states in D_λ are obtained by the action of the group G on the lowest weight state $|- \lambda\rangle$, which can be written explicitly as

$$|\zeta, \lambda\rangle \equiv \mathcal{N}(\zeta) \exp(\zeta_\alpha E_\alpha) |- \lambda\rangle. \quad (15)$$

Here, $\mathcal{N}(\zeta)$ is the normalization constant and α runs over all positive roots.

Next we consider the moment with a positive integer index q

$$M_{|\varphi\rangle}^{(q)} = c_\lambda \int d\mu(\zeta) |\langle \zeta, \lambda | \varphi \rangle|^q. \quad (16)$$

Here, $d\mu(\zeta)$ is the Haar measure of G and $c_\lambda = \dim D_\lambda$ is the normalization constant. The moment can be rewritten as

$$M_{|\varphi\rangle}^{(q)} = c_\lambda \int d\mu(\zeta) (\langle \varphi |^{\otimes q}) (|\zeta, \lambda\rangle^{\otimes q}) (\langle \zeta, \lambda |^{\otimes q}) (\varphi)^{\otimes q}. \quad (17)$$

The important point to note is that the tensor product of the coherent state $|\zeta\rangle^{\otimes q}$ is again a coherent in another irreducible representation $D_{q\lambda} \subset D_\lambda^{(q)}$ whose lowest weight is $-q\lambda$. It can be shown explicitly as

$$|\zeta, \lambda\rangle^{\otimes q} = \mathcal{N}(\zeta)^q \exp(\zeta_\alpha E_\alpha^{(q)}) |- \lambda\rangle^{\otimes q} = |\zeta, q\lambda\rangle. \quad (18)$$

Here, $E_\alpha^{(q)}$ is the representation of E_α in $D_\lambda^{(q)}$, whose explicit form is

$$E_\alpha^{(q)} = E_\alpha \otimes I \otimes \cdots \otimes I + I \otimes E_\alpha \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes E_\alpha. \quad (19)$$

It is easy to verify that $|- \lambda\rangle^{\otimes q}$ is the lowest weight state of $D_{q\lambda}$.

Since the “resolution of unity”

$$I = c_{q\lambda} \int d\mu(\zeta) |\zeta, q\lambda\rangle \langle \zeta, q\lambda| \quad (20)$$

holds in $D_{q\lambda}$, (17) can be written as

$$M_{|\varphi\rangle}^{(q)} = \frac{\dim D_\lambda}{\dim D_{q\lambda}} (\langle \varphi |^{\otimes q}) P_{D_{q\lambda}} (|\varphi\rangle^{\otimes q}), \quad (21)$$

where $P_{D_{q\lambda}}$ is the projection operator to $D_{q\lambda}$. For our purpose, it is convenient to introduce normalized moments

$$\tilde{M}_{|\varphi\rangle}^{(q)} \equiv \frac{\dim D_{q\lambda}}{\dim D_\lambda} M_{|\varphi\rangle}^{(q)} = \langle P_{D_{q\lambda}} \rangle, \quad (22)$$

where the symbol $\langle \rangle$ denotes the expectation value for $|\varphi\rangle^{\otimes q}$. Then $M_{|\varphi\rangle}^{(q)} = 1$ if $|\varphi\rangle$ is a coherent state, and $M_{|\varphi\rangle}^{(q)} < 1$ if it is not.

If the state is expressed as

$$|\varphi\rangle = \sum_i c_i |i\rangle \quad (23)$$

with an orthonormal basis $\{|i\rangle\}$ in D_λ , the normalized q -th moment is

$$\tilde{M}_{|\varphi\rangle}^{(q)} = \sum_j |B_j|^2, \quad (24)$$

$$B_j = \sum_{i_1, \dots, i_q} \langle j, q\lambda | i_1, \dots, i_q \rangle c_{i_1} \dots c_{i_q}, \quad (25)$$

where $\{|j, q\lambda\rangle\}$ is an orthonormal basis of $D_{q\lambda}$, and

$$\langle j, q\lambda | i_1, \dots, i_q \rangle \equiv \langle j, q\lambda | (|i_1\rangle \otimes |i_2\rangle \otimes \dots \otimes |i_q\rangle). \quad (26)$$

Thus we have obtained the explicit formula for the moments.

Since the terms in the sum (25) are invariant under permutations of i_1, i_2, \dots, i_q , the same term appears $q!$ times if all indices are different. In general, the same indices appears in the set $\{i_s\}$. We represent it as $(i_1^{k_1}, i_2^{k_2}, \dots, i_l^{k_l})$, which means the index i_l appears k_l times. Then this term appears $\binom{q}{k_1, k_2, \dots, k_l}$ times in the sum (25), where

$$\binom{q}{k_1, k_2, \dots, k_l} \equiv \begin{cases} \frac{q!}{k_1! k_2! \dots k_l!} & (\text{if } k_1 + k_2 + \dots + k_l = q) \\ 0 & (\text{else}) \end{cases} \quad (27)$$

is the multinomial coefficient. Therefore (25) can be rewritten as

$$B_j = \sum \binom{q}{k_1, k_2, \dots, k_l} \langle j, q\lambda | i_1^{k_1}, i_2^{k_2}, \dots, i_l^{k_l} \rangle c_{i_1}^{k_1} c_{i_2}^{k_2} \dots c_{i_l}^{k_l}, \quad (28)$$

where

$$\langle j, q\lambda | i_1^{k_1}, i_2^{k_2}, \dots, i_l^{k_l} \rangle \equiv \langle j, q\lambda | (|i_1\rangle^{\otimes k_1} \otimes |i_2\rangle^{\otimes k_2} \otimes \dots \otimes |i_l\rangle^{\otimes k_l}). \quad (29)$$

The sum in (28) is taken over all possible combinations of $\{i_s\}$ and $\{k_s\}$.

Let the irreducible decomposition of $D_\lambda^{\otimes q}$ be

$$D_\lambda^{\otimes q} = \sum_{\nu, \tau} D_{\nu, \tau_\nu}. \quad (30)$$

Here, D_{ν, τ_ν} is an irreducible representation with the lowest weight $-\nu$, and the additional integer suffix τ_ν is the multiplicity label. Then

$$\sum_{\nu, \tau_\nu} \langle P_{\nu, \tau_\nu} \rangle = \mathcal{N}^{2q} \quad (31)$$

holds, where P_{ν, τ_ν} is the projection operator to D_{ν, τ_ν} and $\mathcal{N} = (\sum_i |c_i|^2)^{1/2}$ is the norm of $|\varphi\rangle$ which is usually normalized to 1. Hence

$$\tilde{M}_{|\varphi\rangle}^{(q)} = 1 - \sum_{\nu \neq q\lambda, \tau_\lambda} \langle P_{\nu, \tau_\nu} \rangle. \quad (32)$$

Note that $D_{q\lambda}$ is multiplicity-free. In many cases, this formula is more useful and informative than calculating $\langle P_{q\lambda} \rangle$ directly, as we will see later.

III. SIMPLE EXAMPLE: BOSONS IN TWO SINGLE-PARTICLE STATES

A. Algebra of generators

Let us consider a system of m bosons in two single-particle states as a simple example. In this case, there are $m + 1$ many-body states, which are $|0, m\rangle, |1, m - 1\rangle, \dots, |m, 0\rangle$ in the occupation number representation. The many-body states are generated by applying particle-hole excitation operators

$$X_i^j = a_i a_j^\dagger \quad (1 \leq i, j, \leq 2) \quad (33)$$

to the “vacuum” $|0, m\rangle$. Since the total number operator $\hat{N} = a_1^\dagger a_1 + a_2^\dagger a_2$ corresponds to the irrelevant total phase, we can remove it from the algebra. The rest forms the Lie algebra of $SU(2)$:

$$J_+ = a_1 a_2^\dagger, \quad (34)$$

$$J_- = a_2 a_1^\dagger, \quad (35)$$

$$J_z = \frac{1}{2}(a_2 a_2^\dagger - a_1 a_1^\dagger). \quad (36)$$

Actually, it is easy to verify the following commutation relations

$$[J_z, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_z. \quad (37)$$

The two single-particle states $|0, 1\rangle$ and $|1, 0\rangle$ form the fundamental (spin $1/2$) representation of $SU(2)$. The m -particle states form an irreducible representation with spin $m/2$, which we denote as $D_{m/2}$. Hereafter, we mainly use spin quantum numbers j and j_z , instead of occupation numbers, to specify a quantum state. We put suffixes n and s to distinguish the two notations. The relation between the two notations is

$$|n_1, n_2\rangle_n = \left| j = \frac{n_1 + n_2}{2}, j_z = \frac{n_1 - n_2}{2} \right\rangle_s. \quad (38)$$

B. Coherent state and the Husimi distribution

The $SU(2)$ coherent states in the spin j representation are defined as [3]

$$|j, \zeta\rangle = \mathcal{N}(\zeta) \exp(\zeta J_+) |j, -j\rangle_s, \quad (39)$$

where $\mathcal{N}(\zeta)$ is a normalization factor. In our case, the total spin j is determined by the number of particles by $j = m/2$.

ζ is considered to be a coordinate system of $\mathbb{CP}^1 \simeq S^2$, which is related to the angular variables of the sphere (θ, φ) by

$$\zeta = -e^{-i\phi} \tan \frac{\theta}{2}. \quad (40)$$

The coherent state $|j, \zeta\rangle$ is expanded as

$$|j, \zeta\rangle = \sum_{\mu=-j}^j u_{j,\mu}(\zeta) |j, \mu\rangle_s, \quad (41)$$

where

$$u_{j,\mu}(\zeta) = \binom{2j}{j+\mu}^{1/2} e^{-i(j+\mu)\phi} \left(-\sin \frac{\theta}{2} \right)^{j+\mu} \left(\cos \frac{\theta}{2} \right)^{j-\mu}. \quad (42)$$

The Husimi distribution function of a m -particle state $|\varphi\rangle$ is defined as

$$\mathcal{H}_{|\varphi\rangle}(\zeta) = |\langle j = m/2, \zeta | \varphi \rangle|^2. \quad (43)$$

For example, let us consider the simplest nontrivial case $m = 2$ which corresponds to the spin 1 representation. The Husimi distributions of the three basis states are

$$\mathcal{H}_{|2,0\rangle_n}(\theta, \phi) = \sin^4 \frac{\theta}{2}, \quad (44)$$

$$\mathcal{H}_{|1,1\rangle_n}(\theta, \phi) = 2 \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2} = \frac{1}{2} \sin^2 \theta, \quad (45)$$

$$\mathcal{H}_{|0,2\rangle_n}(\theta, \phi) = \cos^4 \frac{\theta}{2}, \quad (46)$$

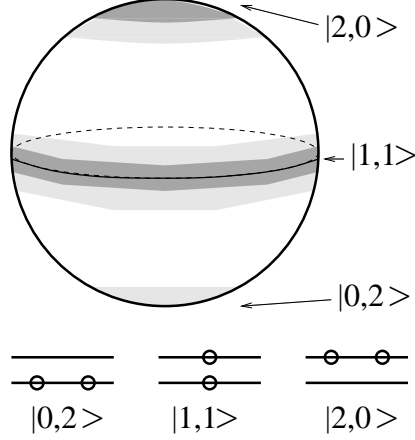


FIG. 2: Rough sketch of the Husimi distribution for bosons with $N = m = 2$. $|2,0\rangle_n$ and $|0,2\rangle_n$ are coherent states, but $|1,1\rangle_n$ is not. Therefore $|1,1\rangle_n$ has a rather extended distribution along the equator, whose normalized second moment is the minimum value $2/3$. This result shows that $|1,1\rangle_n$ can not be written as a tensor product of single-particle states.

which are localized around the north pole ($\theta = \pi$), the equator ($\theta = \pi/2$) and the south pole ($\theta = 0$) respectively. (See Fig. 2.) Among the three states, $|2,0\rangle_n$ and $|0,2\rangle_n$ are separable, but

$$|1,1\rangle_n = \frac{1}{\sqrt{2}}(|1,0\rangle_n|0,1\rangle_n + |0,1\rangle_n|1,0\rangle_n) \quad (47)$$

is not. Corresponding to this fact, the Husimi distribution of $|1,1\rangle_n$ is broader than the other two.

C. moments

The q -th moment of this distribution is defined as

$$M_{|\varphi\rangle}^{(q)} = (2j+1) \int d\mu(\zeta) \{ \mathcal{H}_{|\varphi\rangle}(\zeta) \}^q, \quad (48)$$

where $d\mu(\xi)$ is the Haar measure

$$d\mu(\xi) = \frac{1}{4\pi} \sin \theta d\theta d\phi. \quad (49)$$

$M_{|\varphi\rangle}^{(q)}$ is invariant under the $SU(2)$ transformation by definition of the Haar measure.

We can calculate the moments according to the general formula (24) and (25). Here we consider the second moment, which can be calculated from the CG (Clebsch-Gordan) series

$$D_{m/2} \otimes D_{m/2} = D_m \oplus D_{m-1} \oplus \cdots \oplus D_0. \quad (50)$$

If the state $|\varphi\rangle$ is represented as

$$|\varphi\rangle = \sum_{\nu=-m/2}^{m/2} c_\nu |m/2, \nu\rangle_s, \quad (51)$$

the normalized second moment is

$$\tilde{M}^{(2)} = \frac{2m+1}{m+1} M^{(2)} = \sum_{\mu=-m}^m |B_\mu|^2, \quad (52)$$

$$B_\mu = \langle m, \mu | m/2, m/2; \nu, \nu' \rangle c_\nu c_{\nu'}. \quad (53)$$

Here, $\langle m, \mu | m/2, m/2; \nu, \nu' \rangle$ is the CG coefficient, whose explicit form is

$$\langle 2j, \mu | j, j; \nu, \nu' \rangle = \sqrt{\frac{\binom{2j}{j+\nu} \binom{2j}{j+\nu'}}{\binom{4j}{2j+\mu}}} \delta_{\mu, \nu+\nu'}. \quad (54)$$

Let us examine some simple cases. If $m = 1$ ($j = 1/2$),

$$\tilde{M}_{|\varphi}^{(2)} = \langle P_1 \rangle = (|c_{1/2}|^2 + |c_{-1/2}|^2)^2 = 1, \quad (55)$$

where P_j denotes the projection operator to the spin j representation. This result is natural because this is a one-body system and hence there is no many-body correlation. In the simplest non-trivial case $m = 2$ ($j = 1$), the normalized second moment is

$$\tilde{M}^{(2)} = \langle P_2 \rangle = |c_1|^4 + |c_{-1}|^4 + 2|c_1 c_0|^2 + 2|c_0 c_{-1}|^2 + \frac{2}{3}|c_1 c_{-1} + c_0^2|^2. \quad (56)$$

We can also write it as

$$\tilde{M}^{(2)} = 1 - \langle P_1 \rangle - \langle P_0 \rangle \quad (57)$$

according to (32). $\langle P_1 \rangle$ vanishes identically because the spin 1 part is anti-symmetric with respect to the exchange of the two components of the tensor product. The spin 0 part is

$$|0, 0\rangle_s = \frac{1}{\sqrt{3}} (|1, 1\rangle_s |1, -1\rangle_s - |1, 0\rangle_s |1, 0\rangle_s + |1, -1\rangle_s |1, 1\rangle_s). \quad (58)$$

Hence

$$\tilde{M}_{|\varphi\rangle}^{(2)} = 1 - \frac{1}{3}|c_0^2 - 2c_1c_{-1}|^2. \quad (59)$$

The maximum value of $\tilde{M}^{(2)}$ is 1, which is obtained when $c_0^2 = 2c_1c_{-1}$. This is the condition for the coherent states. The minimum value is 2/3, which is obtained if and only if

$$\Im(c_0^2c_1^*c_{-1}^*) = 0, \quad (60)$$

$$\Re(c_0^2c_1^*c_{-1}^*) \leq 0, \quad (61)$$

$$|c_1| = |c_{-1}|. \quad (62)$$

For example, $|1, 1\rangle_n$ satisfies this condition.

IV. GENERAL TREATMENT OF BOSONIC SYSTEMS

In this section we treat bosonic systems generally.

A. Coherent states for bosonic systems

Suppose there are m bosons in N single-particle states. The particle-hole excitation operators

$$X_i^j = a_i a_j^\dagger \quad (1 \leq i, j \leq N) \quad (63)$$

satisfy the commutation relations of $\mathfrak{u}(N)$:

$$[X_i^j, X_l^k] = X_l^j \delta_i^k - X_i^k \delta_l^j. \quad (64)$$

These operators generates the single-particle transformation:

$$[X_i^j, a_k] = \delta_k^j a_i. \quad (65)$$

The N single-particle states form the fundamental representation of $U(N)$, and the bosonic many-body states form an irreducible representation which consists of symmetric combinations of the single-particle states. In the Young diagram, this representation is $[1^m]$ (See Fig. 3), and we denote it as $B[N, m]$. We will drop $[N, m]$ when it is obvious.

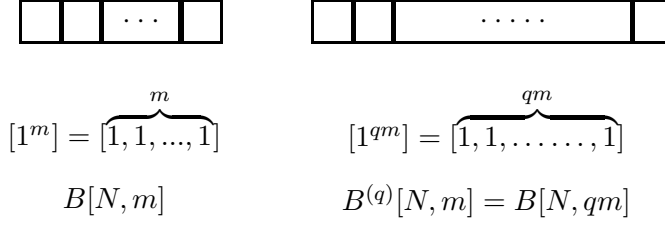


FIG. 3: Irreducible representations of $U(N)$, $B[N, m] = B^{(1)}[N, m]$ and $B^{(q)}[N, m]$.

A coherent state is obtained by applying the $U(N)$ transformation to the bosonic “vacuum” $|0\rangle = |0, \dots, 0, m\rangle$. As we have explained in the introduction, the bosonic coherent states are of the form

$$|\zeta\rangle = |\varphi(\zeta)\rangle |\varphi(\zeta)\rangle \dots |\varphi(\zeta)\rangle \quad (66)$$

where $|\varphi(\zeta)\rangle = U(\zeta)|\varphi_1\rangle$. Unitary transformations in the subspace spanned by $|\varphi_2\rangle, \dots, |\varphi_N\rangle$ is irrelevant for the definition of the coherent state (66), and the phase transformation of $|\varphi_1\rangle$ is also irrelevant. Therefore the coherent states are specified by a point on the manifold

$$U(N)/(U(1) \times U(N-1)) \simeq \mathbb{C}P^{N-1}. \quad (67)$$

An explicit form of the parametrization of the coherent states is

$$|\zeta\rangle = \mathcal{N}(\zeta) \exp\left(\sum_{j=2}^N \zeta_j X_1^j\right) |0\rangle, \quad (68)$$

where $\zeta = (\zeta_2, \zeta_3, \dots, \zeta_N)$ is a coordinate system of $\mathbb{C}P^{N-1}$.

The stationary phase condition of the coherent state path integral leads to an equation of motion on the phase space $\mathbb{C}P^{N-1}$, which describes the dynamics of the condensed bosons.

B. Moments of the Husimi distribution

The Husimi distribution is defined as

$$\mathcal{H}_{|\varphi\rangle}(\zeta) = |\langle \zeta | \varphi \rangle|^2. \quad (69)$$

The normalized q -th moment thereof is

$$\tilde{M}_{|\varphi\rangle}^{(q)} = \frac{\dim D_{q\lambda}}{\dim D_\lambda} M_{|\varphi\rangle}^{(q)} = (\dim D_{q\lambda}) \int d\mu(\zeta) \{\mathcal{H}_{|\varphi\rangle}(\zeta)\}^q. \quad (70)$$

Here, D_λ is the bosonic Hilbert space $B[N, m] = [1^m]$, and $D_{q\lambda}$ corresponds to $[1^{qm}]$, which we denote as $B^{(q)}[N, m]$. (See Fig. 3.) Its dimension is

$$\dim B^{(q)}[N, m] = \frac{(N + qm - 1)!}{(N - 1)!(qm)!}, \quad (71)$$

hence

$$\frac{\dim D_{q\lambda}}{\dim D_\lambda} = \frac{\Gamma(N + qm)\Gamma(m + 1)}{\Gamma(N + m)\Gamma(qm + 1)}. \quad (72)$$

We can calculate the normalized moments by using the general formulae (24) and (25). Let us consider the simplest case $q = 2$. The basis states of $B^{(2)}$ can be constructed as

$$|B^{(2)}, \mathbf{k}\rangle = C \prod_{j=2}^N \left\{ X_1^{(2)j} \right\}^{k_j} |0\rangle_m \otimes |0\rangle_m. \quad (73)$$

Here, $\mathbf{k} = (k_1, k_2, \dots, k_N)$ represents the occupation numbers, where the total number $|\mathbf{k}| \equiv \sum_j k_j$ is constrained to $2m$. C is a normalization constant, $|0\rangle_m \equiv |0, \dots, 0, m\rangle$ is the m -particle vacuum, and

$$X^{(2)j}_i \equiv X_i^j \otimes I + I \otimes X_i^j \quad (74)$$

is a generator of the Lie algebra in $B^{\otimes 2}$. By expanding (73), we obtain

$$|B^{(2)}, \mathbf{k}\rangle = C \prod_{j=2}^N \left\{ \sum_{n_j=0}^{k_j} \binom{k_j}{n_j} (X_1^j)^{n_j} \otimes (X_1^j)^{k_j-n_j} \right\} |0\rangle_m \otimes |0\rangle_m \quad (75)$$

$$= C \sum_{n_2=0}^{k_1} \dots \sum_{n_N=0}^{k_N} \left\{ \prod_{j=2}^N \binom{k_j}{n_j} (X_1^j)^{n_j} \otimes (X_1^j)^{k_j-n_j} \right\} |0\rangle_m \otimes |0\rangle_m, \quad (76)$$

where

$$\prod_{j=2}^N (X_1^j)^{n_j} |0\rangle_m = \left(a_N^\dagger \right)^{n_N} \left(a_{N-1}^\dagger \right)^{n_{N-1}} \dots \left(a_2^\dagger \right)^{n_2} a_1^{n_N+\dots+n_2} \frac{(a_1^\dagger)^m}{\sqrt{m!}} |0\rangle_0 \quad (77)$$

$$= \frac{1}{\sqrt{m!}} \frac{m!}{n_1!} \prod_{j=1}^N \left(a_j^\dagger \right)^{n_j} |0\rangle_0 \quad (78)$$

$$= \frac{\sqrt{m!}}{n_1!} \sqrt{\mathbf{n}!} |\mathbf{n}\rangle. \quad (79)$$

Here, $\mathbf{n} = (n_1, \dots, n_N)$ represents the occupation numbers, where $n_1 \equiv m - (n_N + \dots + n_2)$.

The factorial of the vector \mathbf{n} means

$$\mathbf{n}! \equiv \prod_{j=1}^N n_j!. \quad (80)$$

By substituting (79) into (76), we obtain

$$|B^{(2)}, \mathbf{k}\rangle = C' \sum_{|\mathbf{n}|=m} \sqrt{\frac{\mathbf{k}!}{\mathbf{n}!(\mathbf{k}-\mathbf{n})!}} |\mathbf{n}\rangle \otimes |\mathbf{k}-\mathbf{n}\rangle. \quad (81)$$

The normalization constant C' is determined by

$$\frac{1}{C'^2} = \sum_{|\mathbf{n}|=m} \frac{\mathbf{k}!}{\mathbf{n}!(\mathbf{k}-\mathbf{n})!} \quad (82)$$

$$= \frac{(2m)!}{(m!)^2}. \quad (83)$$

Hence

$$|B^{(2)}, \mathbf{k}\rangle = \sqrt{\frac{(m!)^2}{(2m)!}} \sum_{|\mathbf{n}|=m} \sqrt{\frac{\mathbf{k}!}{\mathbf{n}!(\mathbf{k}-\mathbf{n})!}} |\mathbf{n}\rangle \otimes |\mathbf{k}-\mathbf{n}\rangle. \quad (84)$$

The cases with larger q can be treated in the same way. The result is

$$|B^{(q)}, \mathbf{k}\rangle = \sqrt{\frac{(m!)^q}{(mq)!}} \sum_{|\mathbf{n}_1|=\dots=|\mathbf{n}_q|=m} \left(\begin{matrix} \mathbf{k} \\ \mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_q \end{matrix} \right)^{1/2} |\mathbf{n}_1\rangle \otimes |\mathbf{n}_2\rangle \otimes \dots \otimes |\mathbf{n}_q\rangle, \quad (85)$$

where

$$\left(\begin{matrix} \mathbf{k} \\ \mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_q \end{matrix} \right) \equiv \begin{cases} \frac{\mathbf{k}!}{\mathbf{n}_1! \mathbf{n}_2! \dots \mathbf{n}_q!} & (\text{if } \mathbf{n}_1 + \mathbf{n}_2 + \dots + \mathbf{n}_q = \mathbf{k}) \\ 0 & (\text{else}) \end{cases}. \quad (86)$$

Suppose a state $|\varphi\rangle$ is represented in the occupation number representation:

$$|\varphi\rangle = \sum_{\mathbf{n}} c_{\mathbf{n}} |\mathbf{n}\rangle. \quad (87)$$

From the general formulae (24) and (25), we obtain

$$\tilde{M}_{|\varphi\rangle}^{(q)} = \sum_{|\mathbf{k}|=mq} |B_{\mathbf{k}}|^2, \quad (88)$$

$$B_{\mathbf{k}} = \sqrt{\frac{(m!)^q}{(mq)!}} \sum_{|\mathbf{n}_1|=\dots=|\mathbf{n}_q|=m} \left(\begin{matrix} \mathbf{k} \\ \mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_q \end{matrix} \right)^{1/2} c_{\mathbf{n}_1} c_{\mathbf{n}_2} \dots c_{\mathbf{n}_q}. \quad (89)$$

V. FERMIONS

In this section we study fermionic systems.

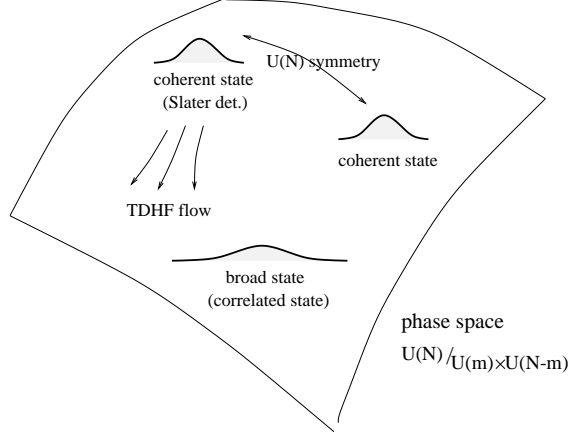


FIG. 4: Schematic picture of many-body Husimi distributions in a fermionic system. The localized wave packets (coherent states) represent independent-particle states (Slater determinants), and broad states represent correlated states. All Slater determinants are connected by the $U(N)$ group, and treated equally. The TDHF equation defines a flow in this phase space.

A. Coherent states for fermionic systems

We consider m fermions in N single-particle states. The particle-hole excitation operators

$$X_i^j = a_i a_j^\dagger \quad (90)$$

satisfy the same equations (64) and (65) as in the bosonic case. Hence they generate the single-particle transformation group $U(N)$. Fermionic many-particle states form an irreducible representation which consists of anti-symmetric combinations of the single-particle states. In the Young diagram, this representation is denoted by $[m]$ (See Fig. 5), and we refer to it as $F[N, m]$.

As we have explained in the introduction, the fermionic coherent states are Slater determinants, which can be written as

$$|F, \zeta\rangle = \mathcal{A}(|\varphi_1(\zeta)\rangle |\varphi_2(\zeta)\rangle \dots |\varphi_m(\zeta)\rangle) \quad (91)$$

where $|\varphi_i(\zeta)\rangle = U(\zeta)|\varphi_i\rangle$. Unitary transformations in the subspaces spanned by $|\varphi_1\rangle \dots |\varphi_m\rangle$ and $|\varphi_{m+1}\rangle \dots |\varphi_N\rangle$ are irrelevant when we consider the coherent states. Therefore the coherent states are specified by a point on the complex Grassmann manifold

$$U(N)/(U(N-m) \times U(m)) \simeq G_{m,N}(\mathbb{C}). \quad (92)$$

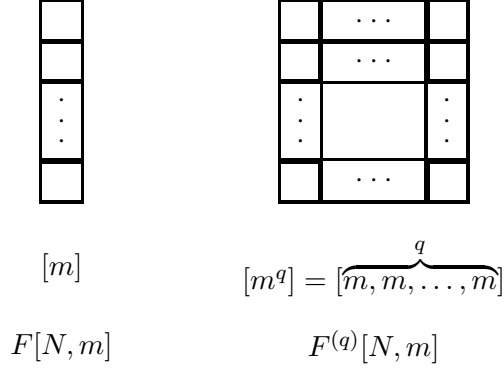


FIG. 5: Irreducible representations of $U(N)$, $F[N, m] = F^{(1)}[N, m]$ and $F^{(q)}[N, m]$.

An explicit parametrization of the coherent states is given by [9]

$$|F, \zeta\rangle = \mathcal{N}(\zeta) \exp \left(\sum_{i=1}^m \sum_{j=m+1}^N \zeta_j^i X_i^j \right) |0\rangle, \quad (93)$$

where $|0\rangle \equiv |\overbrace{0, \dots, 0}^{N-m}, \overbrace{1, \dots, 1}^m\rangle$ is the fermionic “vacuum”, and $\zeta = \{\zeta_j^i\}$ is a coordinate system of $G_{m,N}(\mathbb{C})$.

The “classical” equation of motion defined on the manifold is the TDHF (time-dependent Hartree-Fock) equation [10]. (See Fig. 4.) Therefore the manifold $G_{m,N}(\mathbb{C})$ is sometimes referred to as the TDHF manifold. The Husimi distribution on this manifold is defined as

$$\mathcal{H}_{|\varphi\rangle}(\zeta) = |\langle F, \zeta | \varphi \rangle|^2. \quad (94)$$

B. Moments of the Husimi distribution

Let us consider a m -particle state

$$|\varphi\rangle = \sum_{\mathbf{n}} c_{\mathbf{n}} |\mathbf{n}\rangle, \quad (95)$$

where

$$\mathbf{n} = (n_1, n_2, \dots, n_N) \quad (n_i = 0 \text{ or } 1) \quad (96)$$

and

$$|\mathbf{n}| = \sum_j n_j = m. \quad (97)$$

We calculate the normalized q -th moment of the Husimi distribution

$$\tilde{M}_{|\varphi\rangle}^{(q)} = \frac{\dim D_{q\lambda}}{\dim D_\lambda} M_{|\varphi\rangle}^{(q)}. \quad (98)$$

Here, $D_{q\lambda}$ is $[m^q]$ in the Young diagram, which we refer to as $F^{(q)}[N, m]$ (Fig. 5). In this case,

$$\dim F^{(q)} = \prod_{j=0}^{m-1} \frac{(N+q-j-1)! j!}{(N-j-1)! (q+j)!}, \quad (99)$$

hence

$$\frac{\dim D_{q\lambda}}{\dim D_\lambda} = \prod_{j=0}^{m-1} \frac{\Gamma(N+q-j)\Gamma(j+2)}{\Gamma(N+1-j)\Gamma(q+j+1)}. \quad (100)$$

According to the general formulae (24) and (25),

$$\tilde{M}_{|\varphi\rangle}^{(q)} = \sum_{\mathbf{k}, \tau} \left| B_{\mathbf{k}, \tau} \right|^2, \quad (101)$$

$$B_{\mathbf{k}, \tau} = \sum_{\mathbf{n}} \langle F^{(q)}, \mathbf{k}, \tau | (|\mathbf{n}_1\rangle \otimes \cdots \otimes |\mathbf{n}_q\rangle) c_{\mathbf{n}_1} \cdots c_{\mathbf{n}_q}. \quad (102)$$

Here an additional index τ is introduced because the occupation number \mathbf{k} does not necessarily specify a state in $F^{(q)}$. The group-theoretical factors in (102) can be evaluated, for instance, by the eigenfunction method [11]. However, it is not so easy as in the bosonic case to obtain an explicit formula for them. In the following, we calculate (102) in some simple examples.

C. Examples

Let us consider some simple fermionic systems. The cases with $m = 1$ are trivial, where $\tilde{M}^{(2)}$ is always equal to unity. Let us consider the cases with $m = 2$. It is obviously trivial for $N = 2$. For $N = 3$, there are three many-body states $|011\rangle$, $|101\rangle$ and $|110\rangle$, which form the representation $\bar{\mathbf{3}}$ of $U(3)$ (See Fig. 6.) The representation $F^{(2)}$ is $\bar{\mathbf{6}}$, whose components are written as, for instance,

$$|220\rangle = \begin{vmatrix} 110 \\ 110 \end{vmatrix}, \quad (103)$$

$$|211\rangle = \frac{1}{\sqrt{2}} \left\{ \begin{vmatrix} 110 \\ 101 \end{vmatrix} + \begin{vmatrix} 101 \\ 110 \end{vmatrix} \right\} \quad (104)$$

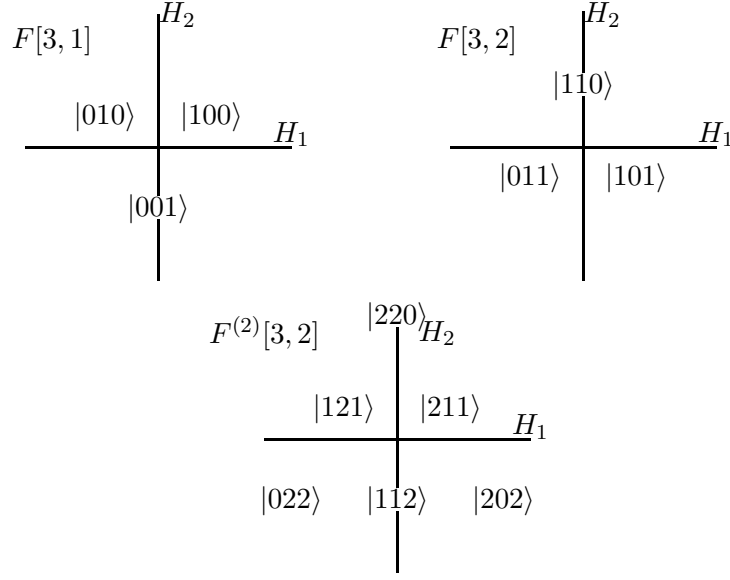


FIG. 6: Irreducible representations of $U(3)$. The weight vector (H_1, H_2) is related to occupation numbers by $H_1 = (n_1 - n_2)/2$ and $H_2 = (n_1 + n_2 - 2n_3)/(2\sqrt{3})$. Single-particle states form the fundamental representation $F[3,1] = \mathbf{3}$, and two-particle (i.e., one-hole) states form $F[3,2] = \bar{\mathbf{3}}$. $F^{(2)}[3,2]$ corresponds to $\bar{\mathbf{6}}$.

Here we arranged the elements of the tensor products vertically. For example,

$$\left| \begin{array}{c} 110 \\ 101 \end{array} \right\rangle \equiv |110\rangle \otimes |101\rangle. \quad (105)$$

Note that the sum of the two row vectors gives the occupation number of the product.

All the components of $\bar{\mathbf{6}}$ are obtained by permutating the occupation numbers in (103) and (104). Then we obtain the components of $\tilde{M}^{(2)}$ as

$$B_{220} = (c_{12})^2, \quad (106)$$

$$B_{211} = \sqrt{2} c_{12} c_{13}, \quad (107)$$

....

Here, c_{ij} is the coefficient of the basis state where i -th and j -th states are occupied. For example, c_{12} is the coefficient of $|011\rangle$. The explicit formula for the second moment is

$$\tilde{M}^{(2)} = (|c_{12}|^2 + |c_{23}|^2 + |c_{31}|^2)^2 = 1. \quad (108)$$

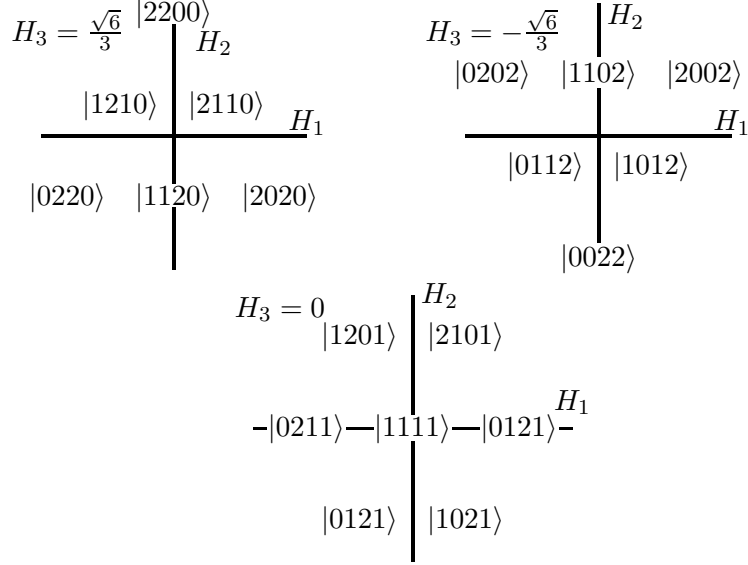


FIG. 7: An irreducible representation of $U(4)$, $F^{(2)}[4, 2] = \mathbf{20}$. The weight vector (H_1, H_2, H_3) is related to occupation numbers by $H_1 = (n_1 - n_2)/2$, $H_2 = (n_1 + n_2 - 2n_3)/(2\sqrt{3})$ and $H_3 = (n_1 + n_2 + n_3 - 3n_4)/(2\sqrt{6})$. This representation forms a regular octahedron in the three-dimensional weight space. $|1111\rangle$ is at the center thereof, and doubly degenerate.

This result is natural because $F[3, 2]$ is a one-hole system, and has no many-body correlation essentially. By the same reason, $F[N, N - 1]$ is also trivial for arbitrary N . In this one-hole case, a state can be written as

$$|\varphi\rangle = \sum_{j=1}^N c_j |j\rangle. \quad (109)$$

Here, $|j\rangle$ is a state with a hole in the j -th single-particle state. Since this system is essentially the same as the one boson system $B[N, 1]$, we can use the general formulae for bosons (89). Then, by using (28) we obtain

$$B_{\mathbf{k}} = \sqrt{\frac{q!}{k_1! k_2! \dots k_N!}} c_1^{k_1} c_2^{k_2} \dots c_N^{k_N} \quad (110)$$

and

$$\tilde{M}^{(q)} = \left(\sum_j |c_j|^2 \right)^q = 1. \quad (111)$$

The simplest non-trivial example is $F[4, 2]$, whose dimension is six. To calculate the

second moment, we have to consider the CG series

$$\begin{aligned} [2] \otimes [2] &= [2, 2] \oplus [3, 1] \oplus [4] \\ \mathbf{6} \otimes \mathbf{6} &= \mathbf{20} \oplus \mathbf{15} \oplus \mathbf{1}. \end{aligned} \quad (112)$$

The components of $F^{(2)} = [2, 2]$ are written as, for instance,

$$|2200\rangle = \begin{vmatrix} 1100 \\ 1100 \end{vmatrix}, \quad (113)$$

$$|2110\rangle = \frac{1}{\sqrt{2}} \left\{ \begin{vmatrix} 1010 \\ 1100 \end{vmatrix} + \begin{vmatrix} 1010 \\ 1100 \end{vmatrix} \right\}. \quad (114)$$

Note that $|1111\rangle$ is doubly degenerate. (See Fig. 7.) We define two basis vectors as

$$|1111, a\rangle = \frac{1}{2} \left\{ \begin{vmatrix} 1100 \\ 0011 \end{vmatrix} + \begin{vmatrix} 0011 \\ 1100 \end{vmatrix} + \begin{vmatrix} 1010 \\ 0101 \end{vmatrix} + \begin{vmatrix} 0101 \\ 1010 \end{vmatrix} \right\}, \quad (115)$$

$$\begin{aligned} |1111, b\rangle &= \frac{1}{2\sqrt{3}} \left\{ \begin{vmatrix} 1100 \\ 0011 \end{vmatrix} + \begin{vmatrix} 0011 \\ 1100 \end{vmatrix} - \begin{vmatrix} 1010 \\ 0101 \end{vmatrix} - \begin{vmatrix} 0101 \\ 1010 \end{vmatrix} \right. \\ &\quad \left. - 2 \begin{vmatrix} 1001 \\ 0110 \end{vmatrix} - 2 \begin{vmatrix} 0110 \\ 1001 \end{vmatrix} \right\}. \end{aligned} \quad (116)$$

Then

$$B_{2200} = (c_{12})^2, \quad (117)$$

$$B_{2110} = \sqrt{2} c_{12} c_{13}, \quad (118)$$

$$B_{1111, a} = c_{12} c_{34} + c_{13} c_{24}, \quad (119)$$

$$B_{1111, b} = \frac{c_{12} c_{34} - c_{13} c_{24} - 2c_{14} c_{23}}{\sqrt{3}}. \quad (120)$$

The other components are obtained by permutating the occupation numbers. Then we obtain

$$\begin{aligned} \tilde{M}^{(2)} &= |c_{12}|^4 + |c_{13}|^4 + |c_{14}|^4 + |c_{23}|^4 + |c_{24}|^4 + |c_{34}|^4 \\ &\quad + 2 \{ |c_{12} c_{13}|^2 + |c_{12} c_{14}|^2 + |c_{13} c_{14}|^2 + |c_{12} c_{23}|^2 + |c_{12} c_{24}|^2 + |c_{23} c_{24}|^2 \\ &\quad + |c_{13} c_{23}|^2 + |c_{13} c_{34}|^2 + |c_{23} c_{34}|^2 + |c_{14} c_{24}|^2 + |c_{14} c_{34}|^2 + |c_{24} c_{34}|^2 \} \\ &\quad + \frac{2}{3} \{ |c_{12} c_{34} + c_{13} c_{24}|^2 + |c_{12} c_{34} - c_{14} c_{23}|^2 + |c_{13} c_{24} + c_{14} c_{23}|^2 \} \end{aligned} \quad (121)$$

A simpler expression of the second moment can be obtained from (32):

$$\tilde{M}^{(2)} = 1 - \langle P_{[3,1]} \rangle - \langle P_{[4]} \rangle. \quad (122)$$

$\langle P_{[3,1]} \rangle$ vanishes identically because it is anti-symmetric with respect to the exchange of the components of the tensor product. Therefore all we have to calculate is $\langle P_{[4]} \rangle$. The basis state of the singlet representation $[4]$ is

$$|[4]\rangle = \frac{1}{\sqrt{6}} \left\{ \begin{vmatrix} 1100 \\ 0011 \end{vmatrix} + \begin{vmatrix} 0011 \\ 1100 \end{vmatrix} - \begin{vmatrix} 1010 \\ 0101 \end{vmatrix} - \begin{vmatrix} 0101 \\ 1010 \end{vmatrix} + \begin{vmatrix} 1001 \\ 0110 \end{vmatrix} + \begin{vmatrix} 0110 \\ 1001 \end{vmatrix} \right\}. \quad (123)$$

Hence

$$\tilde{M}^{(2)} = 1 - \langle P_{[4]} \rangle = 1 - \frac{2}{3} |c_{12}c_{34} - c_{13}c_{24} + c_{14}c_{23}|^2. \quad (124)$$

The second moment of the general two-fermion case $F[N, 2]$ ($N \geq 4$) is obtained by a similar calculation. The result is

$$\tilde{M}^{(2)} = \sum_{i_1, i_2} |c_{i_1, i_2}|^4 + 2 \sum_{j_1, j_2, k} |c_{k, j_1} c_{k, j_2}|^2 + \sum_{l_1, l_2, l_3, l_4} A(l_1, l_2, l_3, l_4). \quad (125)$$

Here,

$$A(i, j, k, l) \equiv \frac{2}{3} \{ |c_{ij}c_{kl} + c_{ik}c_{jl}|^2 + |c_{ij}c_{kl} - c_{il}c_{jk}|^2 + |c_{ik}c_{jl} + c_{il}c_{jk}|^2 \}, \quad (126)$$

and the sum is taken over $1 \leq i_1 < i_2 \leq N$, $1 \leq j_1 < j_2 \leq N$, $1 \leq k \leq N$ and $1 \leq l_1 < l_2 < l_3 < l_4 \leq N$. This can be rewritten as

$$\tilde{M}^{(2)} = 1 - \frac{2}{3} \sum_{l_1, l_2, l_3, l_4} |c_{l_1 l_2} c_{l_3 l_4} - c_{l_1 l_3} c_{l_2 l_4} + c_{l_1 l_4} c_{l_2 l_3}|^2 \quad (127)$$

by using the constraint $\sum_{i_1 i_2} |c_{i_1 i_2}|^2 = 1$.

VI. DISTINGUISHABLE PARTICLES

A. General formalism

Although all elementary particles are considered to be bosons or fermions, we often treat particles as distinguishable if the exchange effect is negligible. When we treat distinguishable particles, we can change the single-particle basis of each particle separately. If there

are m particles and the i -th particle can take N_i states, the group of the single-particle transformation is

$$SU(N_1) \times SU(N_2) \times \cdots \times SU(N_m). \quad (128)$$

We denote the Hilbert space of this system as $D[N_1, N_2, \dots, N_m]$, whose dimension is $\prod_{i=1}^m N_i$. It can also be written as $\overbrace{[1] \otimes [1] \otimes \cdots \otimes [1]}^m$, where i -th $[1]$ is the fundamental representation of $SU(N_i)$. A coherent states of this system is a separable (disentangled) state, which is considered to be a point on the “phase space” $\mathbb{C}P^{N_1-1} \times \mathbb{C}P^{N_2-1} \times \cdots \times \mathbb{C}P^{N_m-1}$. The Husimi distribution is defined as a distribution function on this phase space.

B. Moments

The normalized q -th moment is defined as

$$\tilde{M}_{|\varphi\rangle}^{(q)} = \frac{\dim D_{q\lambda}}{\dim D_\lambda} M_{|\varphi\rangle}^{(q)}. \quad (129)$$

Here, $D_{q\lambda} = [1^q] \otimes \cdots \otimes [1^q]$, which we denote as $D^{(q)}[N_1, \dots, N_m]$. We will drop $[N_1, \dots, N_m]$ when it is obvious.

Let us calculate the normalized moment by using the general formulae (24) and (25). If $m = 1$, this is the same as the bosonic case with $m = 1$. A state in $[1^q]$ is represented by occupation numbers $\mathbf{k} = (k_1, \dots, k_{N_1})$, which satisfies $|\mathbf{k}| = q$. Then

$$\langle D^{(q)}, \mathbf{k} | (|\mathbf{n}_1\rangle \otimes \cdots \otimes |\mathbf{n}_q\rangle) = \frac{1}{\sqrt{q!}} \binom{\mathbf{k}}{\mathbf{n}_1, \dots, \mathbf{n}_{N_1}}^{\frac{1}{2}}. \quad (130)$$

Here, $|D^{(q)}, \mathbf{k}\rangle \in [1^q]$ and $|\mathbf{n}_i\rangle \in [1]$. Since $\mathbf{n}_i! = 1$, (130) can be written simply as

$$\langle D^{(q)}, \mathbf{k} | (|\mathbf{n}_1\rangle \otimes \cdots \otimes |\mathbf{n}_q\rangle) = \begin{cases} \sqrt{\frac{\mathbf{k}!}{q!}} & \left(\text{if } \sum_{l=1}^{N_1} \mathbf{n}_l = \mathbf{k} \right) \\ 0 & \text{(else)} \end{cases}. \quad (131)$$

The coefficients in (25) for multi-particle cases are easily obtained from (131). A basis state in $D^{(q)}[N_1, \dots, N_m]$ is specified by a set of occupation numbers $\{k_{i,j_i}\}$, where $1 \leq i \leq m$, $1 \leq j_i \leq N_i$ and $\sum_{j_i} k_{i,j_i} = q$ for $\forall i$. Then we have

$$\langle D^{(q)}, \{k_{i,j_i}\} | (|\{n_{i,j_i,1}\}\rangle \otimes \cdots \otimes |\{n_{i,j_i,q}\}\rangle) = \begin{cases} \sqrt{\frac{\prod_{i,j_i} k_{i,j_i}!}{(q!)^m}} & \left(\text{if } \sum_{l=1}^{N_i} n_{i,j_i,l} = k_{i,j_i} \text{ for } \forall (i, j_i) \right) \\ 0 & \text{(else)} \end{cases} \quad (132)$$

where $|D^{(q)}, \{k_{i,j_i}\}\rangle \in D^{(q)}[N_1, \dots, N_m]$ and $|\{n_{i,j_i,l}\}\rangle \in D[N_1, \dots, N_m]$.

C. Qubits

In this and the next subsections, we investigate systems of qubits rather in detail. Let us consider a system of m -qubits, which is $D[\overbrace{2, 2, \dots, 2}^m]$ in our notation. This is the spin $(1/2, 1/2, \dots, 1/2)$ representation of $SU(2) \times SU(2) \times \dots \times SU(2)$, and $D^{(q)}[2, 2, \dots, 2]$ is the spin $(q/2, q/2, \dots, q/2)$ representation thereof. We denote a basis state of this system as $|\boldsymbol{\mu}\rangle \equiv |\mu_1, \mu_2, \dots, \mu_m\rangle$, where μ_i takes $1/2$ or $-1/2$. This notation is better than using 1 and 0 in that the symmetry between the two states is obvious. In the same way, a basis state in $D^{(q)}[2, 2, \dots, 2]$ is denoted as $|\mu_1, \mu_2, \dots, \mu_m\rangle$, where μ_i runs from $-q/2$ to $q/2$ with the unity steps. This is related to the occupation number representation in the previous subsection by

$$n_{i,1} = \frac{q}{2} + \mu_i, \quad (133)$$

$$n_{i,2} = \frac{q}{2} - \mu_i. \quad (134)$$

According to the results in the previous subsection, the normalized q -th moment

$$\tilde{M}_{|\varphi\rangle}^{(q)} = \left(\frac{q+1}{2}\right)^m M_{|\varphi\rangle}^{(q)} \quad (135)$$

is obtained as

$$\tilde{M}_{|\varphi\rangle}^{(q)} = \sum_{\boldsymbol{\nu}} |B_{\boldsymbol{\nu}}|^2, \quad (136)$$

$$B_{\boldsymbol{\nu}} = \sum_{\boldsymbol{n}_1, \dots, \boldsymbol{n}_q} \langle D^{(q)}, \boldsymbol{\nu} | (|\boldsymbol{\mu}_1\rangle \otimes |\boldsymbol{\mu}_2\rangle \otimes \dots \otimes |\boldsymbol{\mu}_q\rangle) c_{\boldsymbol{\mu}_1} \dots c_{\boldsymbol{\mu}_q} \quad (137)$$

and

$$\langle D^{(q)}, \boldsymbol{\nu} | (|\boldsymbol{\mu}_1\rangle \otimes |\boldsymbol{\mu}_2\rangle \otimes \dots \otimes |\boldsymbol{\mu}_q\rangle) = \begin{cases} \prod_{i=1}^m \left(\frac{q}{2} + \nu_i\right)^{-1/2} & \left(\text{If } \sum_{j=1}^q \boldsymbol{\mu}_j = \boldsymbol{\nu}\right) \\ 0 & (\text{else}) \end{cases}. \quad (138)$$

D. Examples

Let us start with a single qubit case $D[2]$. A qubit state is represented as

$$|\varphi\rangle = c_+|+\rangle + c_-|-\rangle, \quad (139)$$

where $+$ and $-$ are abbreviations of $+1/2$ and $-1/2$. The normalized q -th moment of the state is

$$\tilde{M}_{|\varphi\rangle}^{(q)} = \sum_{\nu=-q/2}^{q/2} |B_\nu|^2, \quad (140)$$

where B_ν is obtained from (28) and (138) as

$$B_\nu = \left(\begin{matrix} q \\ \frac{q}{2} + \nu \end{matrix} \right)^{1/2} c_+^{q/2+\nu} c_-^{q/2-\nu}. \quad (141)$$

Therefore

$$\tilde{M}^{(q)} = (|c_+|^2 + |c_-|^2)^q = 1. \quad (142)$$

This case is trivial, as it should be.

Next we consider 2-qubit case $D[2, 2]$. A state in this space is represented as

$$|\varphi\rangle = c_{++}|++\rangle + c_{+-}|+-\rangle + c_{-+}| - + \rangle + c_{--}| -- \rangle. \quad (143)$$

$D^{(2)}[2, 2]$ is a 9-dimensional representation, whose basis states are specified by $\boldsymbol{\nu} = (\nu_1, \nu_2)$ ($-1 \leq \nu_i \leq 1$). The coefficients $B_{\boldsymbol{\nu}}$ are easily obtained as

$$B_{11} = (c_{++})^2, \quad (144)$$

$$B_{10} = \sqrt{2} c_{++} c_{+-}, \quad (145)$$

$$B_{00} = c_{++} c_{--} + c_{+-} c_{-+}, \quad (146)$$

...

The other components which are not explicitly written here are obtained by exchanging the first and the second suffixes and changing their signs ($+\leftrightarrow -$, $1\leftrightarrow -1$).

The explicit form of the normalized second moment is

$$\begin{aligned} \tilde{M}_{|\varphi\rangle}^{(2)} &= |c_{++}|^4 + |c_{+-}|^4 + |c_{-+}|^4 + |c_{--}|^4 \\ &\quad + 2|c_{++}c_{+-}|^2 + 2|c_{++}c_{-+}|^2 + 2|c_{--}c_{+-}|^2 + 2|c_{--}c_{-+}|^2 \\ &\quad + |c_{++}c_{--} + c_{+-}c_{-+}|^2. \end{aligned} \quad (147)$$

Let us rewrite this by using (32) as

$$\tilde{M}_{|\varphi\rangle}^{(2)} = 1 - \langle P_{(1,0)} \rangle - \langle P_{(0,1)} \rangle - \langle P_{(0,0)} \rangle, \quad (148)$$

where (J_1, J_2) represents the irreducible representation with the total spins J_1 and J_2 . The spin 1 representation, which has three components

$$|1\rangle = \begin{vmatrix} + \\ + \end{vmatrix}, \quad |0\rangle = \frac{1}{\sqrt{2}} \left\{ \begin{vmatrix} + \\ - \end{vmatrix} + \begin{vmatrix} - \\ + \end{vmatrix} \right\}, \quad |-1\rangle = \begin{vmatrix} - \\ - \end{vmatrix}, \quad (149)$$

are symmetric with respect to the exchange between the components of the tensor product. The spin 0 representation, whose basis state is

$$|0\rangle = \frac{1}{\sqrt{2}} \left\{ \begin{vmatrix} + \\ - \end{vmatrix} - \begin{vmatrix} - \\ + \end{vmatrix} \right\}, \quad (150)$$

is anti-symmetric with respect to the exchange. Note that we arranged the components of the tensor product vertically so that we can distinguish it from the tensor product to represent the physical composite systems. $\langle P_{(1,0)} \rangle$ and $\langle P_{(0,1)} \rangle$ vanish because the representations $(1, 0)$ and $(0, 1)$ are anti-symmetric as a whole, and only $\langle P_{(0,0)} \rangle$ survives in (148). The basis state of the singlet representation $(0, 0)$ is

$$|00\rangle = \frac{1}{\sqrt{2}} \left\{ \begin{vmatrix} + \\ - \end{vmatrix} - \begin{vmatrix} - \\ + \end{vmatrix} \right\} \otimes \frac{1}{\sqrt{2}} \left\{ \begin{vmatrix} + \\ - \end{vmatrix} - \begin{vmatrix} - \\ + \end{vmatrix} \right\} \quad (151)$$

$$= \frac{1}{2} \left\{ \begin{vmatrix} ++ \\ -- \end{vmatrix} + \begin{vmatrix} -- \\ ++ \end{vmatrix} - \begin{vmatrix} +- \\ -+ \end{vmatrix} - \begin{vmatrix} -+ \\ +- \end{vmatrix} \right\}. \quad (152)$$

Then we obtain

$$\langle P_{(0,0)} \rangle = |c_{++}c_{--} - c_{+-}c_{-+}|^2 = \frac{C^2}{4}, \quad (153)$$

where C is the concurrence [12]. Hence

$$\tilde{M}_{|\varphi\rangle}^{(2)} = 1 - \frac{C^2}{4}. \quad (154)$$

The next example is the second moment of the 3-qubit case.

$$B_{111} = (c_{+++})^2, \quad (155)$$

$$B_{110} = \sqrt{2}c_{+++}c_{++-}, \quad (156)$$

$$B_{100} = c_{+++}c_{+--} + c_{++-}c_{+-+}, \quad (157)$$

$$B_{000} = \frac{1}{\sqrt{2}}(c_{+++}c_{---} + c_{++-}c_{--+} + c_{+-+}c_{-+-} + c_{-++}c_{+-}). \quad (158)$$

The other components are obtained by permutating the suffixes and changing their signs. Then we obtain

$$\begin{aligned}
\tilde{M}_{|\varphi\rangle}^{(2)} = & |c_{+++}|^4 + |c_{---}|^4 + |c_{++-}|^4 + |c_{+-+}|^4 + |c_{-++}|^4 + |c_{--+}|^4 + |c_{-+-}|^4 + |c_{+--}|^4 \\
& + 2|c_{+++}c_{++-}|^2 + 2|c_{+++}c_{+-+}|^2 + 2|c_{+++}c_{-++}|^2 + 2|c_{---}c_{--+}|^2 \\
& + 2|c_{---}c_{-+-}|^2 + 2|c_{---}c_{+--}|^2 + 2|c_{++-}c_{--+}|^2 + 2|c_{+-+}c_{-+-}|^2 \\
& + 2|c_{++-}c_{-+-}|^2 + 2|c_{-++}c_{+--}|^2 + 2|c_{++-}c_{+--}|^2 + 2|c_{+-+}c_{--+}|^2 \\
& + |c_{+++}c_{+-+} + c_{++-}c_{--+}|^2 + |c_{+++}c_{--+} + c_{+-+}c_{++-}|^2 + |c_{+++}c_{-+-} + c_{--+}c_{+++}|^2 \\
& + |c_{---}c_{++-} + c_{--+}c_{-+-}|^2 + |c_{---}c_{+-+} + c_{-+-}c_{+--}|^2 + |c_{---}c_{+--} + c_{+--}c_{---}|^2 \\
& + \frac{1}{2}|c_{+++}c_{---} + c_{++-}c_{--+} + c_{+-+}c_{-+-} + c_{-++}c_{+--}|^2.
\end{aligned} \tag{159}$$

By using (32), $\tilde{M}^{(2)}$ can also be written as

$$\tilde{M}^{(2)} = 1 - \langle P_{(0,0,1)} \rangle - \langle P_{(0,1,0)} \rangle - \langle P_{(1,0,0)} \rangle. \tag{160}$$

Here we dropped the anti-symmetric representations $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$ and $(0, 0, 0)$. $(0, 0, 1)$ is a 3-dimensional representation, whose basis states are

$$|001\rangle = \frac{1}{2} \left\{ \begin{vmatrix} + & + & + \\ - & - & + \end{vmatrix} + \begin{vmatrix} - & - & + \\ + & + & + \end{vmatrix} - \begin{vmatrix} + & - & + \\ - & + & + \end{vmatrix} - \begin{vmatrix} - & + & + \\ + & - & + \end{vmatrix} \right\}, \tag{161}$$

$$\begin{aligned}
|000\rangle = & \frac{1}{2\sqrt{2}} \left\{ \begin{vmatrix} + & + & + \\ - & - & - \end{vmatrix} + \begin{vmatrix} - & - & + \\ + & + & - \end{vmatrix} - \begin{vmatrix} + & - & + \\ - & + & - \end{vmatrix} - \begin{vmatrix} - & + & + \\ + & - & - \end{vmatrix} \right. \\
& \left. + \begin{vmatrix} + & + & - \\ - & - & + \end{vmatrix} + \begin{vmatrix} - & - & - \\ + & + & + \end{vmatrix} - \begin{vmatrix} + & - & - \\ - & + & + \end{vmatrix} - \begin{vmatrix} - & + & - \\ + & - & + \end{vmatrix} \right\},
\end{aligned} \tag{162}$$

$$|00-1\rangle = \frac{1}{2} \left\{ \begin{vmatrix} + & + & - \\ - & - & - \end{vmatrix} + \begin{vmatrix} - & - & - \\ + & + & - \end{vmatrix} - \begin{vmatrix} + & - & - \\ - & + & - \end{vmatrix} - \begin{vmatrix} - & + & - \\ + & - & - \end{vmatrix} \right\}. \tag{163}$$

Hence

$$\langle P_{(0,0,1)} \rangle = \frac{1}{2}|c_{+++}c_{--+} - c_{++-}c_{+-+}|^2 + |c_{++-}c_{---} - c_{+-+}c_{-+-}|^2 \tag{164}$$

$$+ \frac{1}{2}|c_{+++}c_{---} + c_{++-}c_{--+} - c_{+-+}c_{-+-} - c_{-++}c_{+--}|^2. \tag{165}$$

By explicit calculation this is shown to be equal to $\frac{1}{4}\text{Tr}(\rho_{AB}\tilde{\rho}_{AB})$. Here, A,B and C are the three qubits. ρ_{AB} is the density matrix of the pair A and B which is obtained by tracing

out the information of C. $\tilde{\rho}_{AB}$ is the “spin-flipped” density matrix [13]

$$\tilde{\rho}_{AB} = (\sigma_y \otimes \sigma_y) \rho_{AB}^* (\sigma_y \otimes \sigma_y), \quad (166)$$

where the asterisk denotes complex conjugation in the standard basis and

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (167)$$

According to [14],

$$\text{Tr}(\rho_{AB} \tilde{\rho}_{AB}) = C_{AB}^2 + \frac{1}{2} \tau_{ABC}, \quad (168)$$

where C_{AB} is the concurrence of the density matrix ρ_{AB} and τ_{ABC} is the 3-tangle whose explicit form is given in [14]. In the same way we can calculate $\langle P_{(0,1,0)} \rangle$ and $\langle P_{(0,0,1)} \rangle$, and obtain the result

$$\tilde{M}^{(2)} = 1 - \frac{1}{4}(C_{AB}^2 + C_{BC}^2 + C_{CA}^2) - \frac{3}{8} \tau_{ABC}. \quad (169)$$

VII. CONCLUDING REMARKS

In this paper, we have defined generalized Husimi distributions for many-body systems, and showed that their moments can be practically useful measures of complexity of many-body quantum states. We have derived an algebraic formula to evaluate the moments with integer indices. For bosons and distinguishable particles we have derived the group theoretical factors appearing in the formula explicitly. We have also examined some special cases of fermions, though we have not yet obtained a general formula for the group-theoretical factors in fermionic cases.

The moments take the maximum values if and only if the state is an independent-particle state. Therefore the moments are considered to be measures of entanglement. Although a lot of measures of entanglement have been proposed so far, our method has an advantage of being able to produce infinitely many measures systematically. Of course, not all of them are independent. Nevertheless, it might be possible that the moments and the related invariants (See (32)) form a complete set of algebraic invariants to classify the pure state entanglement.

The relation between delocalization of the Husimi distribution and chaoticity of the classical mechanics has been shown in one-body systems [1, 15]. We can expect a similar

correspondence also in many-body systems, which means a relation between many-body correlation and chaoticity of the dynamics of coherent states (mean fields). For example, chaotic behaviors in fermionic systems like nuclei have been studied from the viewpoints of the shell model [16] and the mean field dynamics [10]. However, the relation between the two standpoints has been unclear. We hope that the Husimi distribution will be a bridge between the two fields.

In this paper, we have concerned only with the systems with definite particle numbers, and considered classifications based on the single-particle unitary transformations. When we treat a systems with variable particle number, we can introduce a generalized one-body transformation, which is known as the Bogoliubov transformation. Then, for example, BCS-type wavefunctions are considered to be coherent states in fermionic systems. The treatment of such systems will be reported elsewhere.

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- [17] The Husimi distribution can be defined more generally as $\mathcal{H}(\mathbf{p}, \mathbf{q}) \equiv \langle \mathbf{p}, \mathbf{q} | \hat{\rho} | \mathbf{p}, \mathbf{q} \rangle$, where $\hat{\rho}$ is the density matrix. However, we restrict ourselves to pure states in this paper.
- [18] Note that the “vacuum” here may be different from the physical vacuum which is defined as the lowest energy state. From the group theoretical point of view, it is convenient to use the lowest (or the highest) weight state as the “vacuum”. See section II.
- [19] It does not matter whether we use $U(N)$ or $SU(N)$, since $U(N) \simeq U(1) \times SU(N)$ and the $U(1)$ subgroup corresponds to the irrelevant phase factor. Nevertheless we prefer $U(N)$ rather than $SU(N)$ for systems of identical particles, because the basis of the Lie algebra of $U(N)$ is easier to write down explicitly. See section IV and V.
- [20] In fact, no state gives such a distribution. This is an example only to explain the meaning of the Rényi-Wehrl entropy.